Anomalous electron trapping by localized magnetic fields

1999 J. Phys. A: Math. Gen. 32 3029

(http://iopscience.iop.org/0305-4470/32/16/011)

J. Phys. A: Math. Gen. **32** (1999) 3029–3039. Printed in the UK PII: S0305-4470(99)00555-7

Anomalous electron trapping by localized magnetic fields

F Bentosela†∥, R M Cavalcanti‡¶, P Exner§+ and V A Zagrebnov†*

† Centre de Physique Théorique, CNRS, 13288 Marseille Luminy, France
and
Université de la Mediterranée (Aix–Marseille II), 13288 Marseille Luminy, France
‡ Instituto de Física, Universidade de São Paulo, Cx. Postal 66318, 05315 - 970 São Paulo, SP, Brazil
§ Nuclear Physics Institute, Academy of Sciences, 25068 Řež, Czech Republic
and
Doppler Institute, Czech Technical University, Břehová 7, 11519 Prague, Czech Republic

Received 22 December 1998

Abstract. We consider an electron with an anomalous magnetic moment $g > 2$ confined to a plane and interacting with a non-zero magnetic field B perpendicular to the plane. We show that if B has compact support and the magnetic flux in natural units is $F \geqslant 0$, the corresponding Pauli Hamiltonian has at least $1 + [F]$ bound states, without making any assumptions about the field profile. Furthermore, in the zero-flux case there is a pair of bound states with opposite spin orientations. Using a Birman–Schwinger technique, we extend the last claim to a weak rotationally symmetric field with $B(r) = \mathcal{O}(r^{-2-\delta})$, thus correcting a recent result. Finally, we show that under mild regularity assumptions existence of the bound states can also be proved for non-symmetric fields with tails.

1. Introduction

The interaction of electrons with a localized magnetic field has been a subject of interest for a long time. It was observed recently that a magnetic flux tube can bind charged particles with anomalous magnetic moment $g > 2$. An example of such a particle is the electron, which has $g = 2.0023$.

The effect was observed first in simple examples [BV, CFC, Mo], such as a cylindrical tube with a field which is either homogeneous or supported by the tube surface. The same behaviour was then demonstrated for any rotationally invariant field $B(r)$ with compact support, and which does not change sign [CC]. In the next step the symmetry condition was removed and the positivity requirement weakened [BEZ2]. The main aim of the present paper is to complete this process by showing that bound states exist for any (non-trivial) compactly supported field and their number is controlled by the number of flux quanta: the corresponding Pauli Hamiltonian will be shown to have at least $1 + [F]$ negative eigenvalues, where F is the value of the flux through the tube in natural units.

This improvement is made possible by a pair of new tools. First of all, the supersymmetry properties of the Pauli operators allow us to show that the matrix element of the field appearing

∥ E-mail address: bentosela@cpt.univ-mrs.fr
¶ E-mail address: rmoritz@fma.if.usp.br
+ E-mail address: exner@ujf.cas.cz
* E-mail address: zagrebnov@cpt.univ-mrs.fr

in the sufficient condition of [BEZ2] in fact has a definite sign. This trick will be combined with a more sophisticated variational estimate, which enables us to treat the integer-flux situation on the same footing as the other cases. In particular, we will be able to demonstrate in this way that for a non-zero B a bound state due to an excess magnetic moment exists even if the flux is zero. More than that, an analogous argument shows that in this situation the field binds electrons with *both* spin orientations.

While the proof of the last result requires a compact support and does not cover fields with tails extending to infinity, it raises a question about a claim made in a recent paper by some of us [BEZ1]. It was said there that a system with a particular rotationally symmetric field induced by an electric current vortex has no bound states for weak currents. This is not correct: the statement is true for higher partial waves only, while the s-wave part in reality has a non-trivial spectrum for any non-zero current.

The error is subtle and—we hope—instructive: it illustrates well the fine nature of weakly bound states of Schrödinger operators in one and two dimensions. The point is that caution is needed when the coupling is switched off *nonlinearly*: the case in question represents an example of a two-dimensional Schrödinger operator with a potential which has a *positive mean* for any non-zero coupling constant while still having a bound state.

To set things straight, in sections 4, 5 and 6 we discuss the weak field, zero-flux case in detail by performing the corresponding Birman–Schwinger analysis to second order. For centrally symmetric fields it yields $g > 2$ as a sufficient condition for the existence of bound states, and provides an asymptotic formula for the bound-state energy. We also show that adding some regularity assumptions one can prove in this way the existence of weakly bound states for non-symmetric fields with tails as well.

2. Preliminaries

As we have said, we consider a particle of spin $\frac{1}{2}$ living in a plane and subject to a non-homogeneous magnetic field B perpendicular to the plane. Here and in the next section we suppose that B has support in a compact region Σ of $\mathbb{R}^2$; later we shall replace this by a suitable decay requirement. No hypotheses are made here about the field profile; we assume only its integrability, $B \in L^1(\Sigma)$. The corresponding vector potential $A = (A_1, A_2)$ lies in the plane and $B = \partial_1 A_2 - \partial_2 A_1$. Throughout the paper we employ natural units, $2m = \overline{h} = c = e = 1$.

Remark. The assumptions do not include the singular field profile $B(x) = 2\pi F\,\delta(x)$ (a magnetic string). Although it can be regarded as a squeezing limit of L^1 fields, the procedure is non-trivial: as pointed out in [BV] one has at the same time to perform the non-physical limit $g \to 2$ to preserve the existence of bound states in analogy with the coupling constant renormalization for the usual two-dimensional δ interaction [AGHH]. We will not discuss this case here.

The particle dynamics is described by the two-dimensional Pauli Hamiltonian which we write in the standard form [Th]:

$$H_P^{(\pm)}(A) = (-\mathrm{i}\nabla - A(x))^2 \pm \tfrac{1}{2} g\, B(x) = D^* D + \tfrac{1}{2}(2 \pm g) B(x) \tag{2.1}$$

where $D := (p_1 - A_1) + \mathrm{i}(p_2 - A_2)$ and the two signs correspond to the two possible spin orientations. The quantity

$$F := \frac{1}{2\pi} \int_\Sigma B(x)\, \mathrm{d}^2 x \tag{2.2}$$

is the total flux measured in the natural units $(2\pi)^{-1}$, or the number of flux quanta through Σ. We assume conventionally that $F \geqslant 0$, i.e. if the mean field is non-zero it points up. In such a case we will be interested primarily in the operator $H_P^{(-)}(A)$ which describes an electron with its magnetic moment parallel to the flux.

Next we have to recall a classical result of Aharonov and Casher [AC, Th], which will be a basic ingredient of our argument in the next section. It states that if $F = N + \varepsilon$, $\varepsilon \in (0, 1]$ for a positive integer N, the operator $H_P^{(-)}(A)$ with non-anomalous moment, $g = 2$, has N zero energy eigenvalues. In the gauge $A_1 = -\partial_2\phi$, $A_2 = \partial_1\phi$, where

$$\phi(x) := \frac{1}{2\pi}\int_\Sigma B(y)\, \ln|x - y|\, \mathrm{d}^2 y \tag{2.3}$$

the corresponding eigenfunctions are given explicitly by

$$\chi_j(x) = \mathrm{e}^{-\phi(x)}\,(x_1 + \mathrm{i}x_2)^j \qquad j = 0, 1, \ldots, N-1. \tag{2.4}$$

It is easy to check that $D\chi_j = 0$ for any non-negative integer j, but only those functions listed in (2.4) are square-integrable; this follows from the fact that $\chi_j(x) = \mathcal{O}(|x|^{-F+j})$ as $|x| \to \infty$ (cf [AC], [Th, section 7.2]). However, the functions χ_j with $j = [F]$ and $j = [F] - 1$ (the latter in the case where F is a positive integer; as usual, the symbol $[\cdot]$ denotes the integer part) are zero energy resonances, since they solve the equation $H_P^{(-)}(A)\chi_j = 0$ and do not grow at large distances.

3. Flux tubes

Now we are in position to state our main result about the existence and number of bound states of the operator (2.1).

Theorem 1. *If $B \in L^1$ is non-zero and compactly supported, the operator $H_P^{(-)}(A)$ has for $g > 2$ at least $1 + [F]$ negative eigenvalues. Moreover, if $F = 0$ then $H_P^{(+)}(A)$ also has a bound state.*

Proof. By the minimax principle, it is sufficient for the first claim to find a subspace of dimension $1 + [F]$ on which the quadratic form

$$(\psi, H_P^{(-)}(A)\psi) = \int_{\mathbb{R}^2} |(D\psi)(x)|^2\, \mathrm{d}^2x - \frac{1}{2}\,(g-2)\int_{\mathbb{R}^2} B(x)|\psi(x)|^2\, \mathrm{d}^2x$$

is negative. To construct appropriate trial functions ψ_α we employ the above mentioned zero-energy solutions; specifically, we choose

$$\psi_\alpha(x) = \sum_{j=0}^{[F]} \alpha_j \left(f_{R,\kappa}(r)\chi_j(x) + \varepsilon h_j(x)\right) \tag{3.1}$$

where $h_j \in C_0^2(\Sigma)$ will be specified later and $f_{R,\kappa} : \mathbb{R}_+ \to \mathbb{R}$ is a suitable function such that $f_{R,\kappa}(r) = 1$ for $r := |x| \leqslant R$, with R chosen in such a way that Σ is a subset of $\mathcal{B}_R := \{x : |x| \leqslant R\}$. Clearly, it is sufficient to consider coefficient vectors $\alpha \in \mathbb{C}^{1+[F]}$ with $|\alpha| = 1$.

It is straightforward to compute the value of the energy form; with a later purpose in mind we write it as

$$(\psi_\alpha, (D^*D + \mu B)\psi_\alpha) = \sum_{j,k=0}^{[F]} \bar{\alpha}_j\alpha_k \left\{ \int_{\mathbb{R}^2} \left|f'_{R,\kappa}(r)\right|^2 (\bar{\chi}_j\chi_k)(x)\, \mathrm{d}^2x \right.$$

$$+\, \varepsilon^2 \int_\Sigma (\overline{Dh_j})(x)(Dh_k)(x)\, \mathrm{d}^2x + \mu\left[\int_\Sigma (B\,\bar{\chi}_j\chi_k)(x)\, \mathrm{d}^2x\right.$$

$$+\varepsilon \int_\Sigma ((\bar{h}_j \chi_k + \bar{\chi}_j h_k)B)(x)\, \mathrm{d}^2x + \varepsilon^2 \int_\Sigma (B\bar{h}_j h_k)(x)\, \mathrm{d}^2x \Bigg]\Bigg\} \tag{3.2}$$

with $\mu = -\frac{1}{2}(g-2)$. We have employed here the property $D\chi_j = 0$ of the AC functions and the fact that h_j and $f'_{R,\kappa}$ have by assumption disjoint supports: we have $D\Sigma_j \alpha_j f_{R,\kappa}\chi_j = 0$ inside Σ so $D\psi_\alpha = \varepsilon \Sigma_j \alpha_j Dh_j$ there, while outside Σ we have $h_j = 0$, so $D\psi_\alpha = Df_{R,\kappa}\Sigma_j \alpha_j \chi_j = \Sigma_j \alpha_j \chi_j (p_1 + ip_2) f_{R,\kappa} = \chi_\alpha(-\mathrm{i}x_1 + x_2)r^{-1} f'_{R,\kappa}$.

As a warming-up exercise, suppose first that $F - j > 1$ holds for all non-zero coefficients α_j. Then the corresponding $\chi_j \in L^2$ and we can use the simplest choice $f_{R,\kappa} = 1$ and $\varepsilon = 0$ in (3.1), obtaining

$$(\psi_\alpha, H_P^{(-)}(A)\psi_\alpha) = -\frac{1}{2}(g-2)\int_\Sigma B(x)|\psi_\alpha(x)|^2\, \mathrm{d}^2x. \tag{3.3}$$

Suppose that $(\psi_\alpha, B\psi_\alpha) \leqslant 0$. Since $D^*D\psi_\alpha = 0$, this would imply the inequality $(\psi_\alpha, (D^*D+2B)\psi_\alpha) \leqslant 0$, but the operator in parenthesis equals DD^* giving thus $\|D^*\psi_\alpha\|^2 \leqslant 0$. This is possible only if $D^*\psi_\alpha = 0$, which is false, because otherwise we would have $2B\psi_\alpha = (DD^* - D^*D)\psi_\alpha = 0$ or $B(x)\psi_\alpha = 0$ almost everywhere. Since ψ_α is a product of a positive function $\mathrm{e}^{-\phi(x)}$ and a polynomial in $x_1 + \mathrm{i}x_2$, it has at most $[F] - 1$ zeros (recall that we are assuming $j < F - 1$) and we arrive at $B(x) = 0$, a.e. which contradicts the assumption. Consequently, the right-hand side of (3.3) is negative for $g > 2$.

If the linear combination includes α_j with $0 \leqslant F - j \leqslant 1$, the situation is more complicated. Since the corresponding AC functions are no longer L^2, we have to modify the trial function at large distances, but to a sufficiently small degree to make the positive energy contribution from the tails small. We achieve that by choosing

$$f_{R,\kappa}(r) := \min\left\{1, \frac{K_0(\kappa r)}{K_0(\kappa R)}\right\} \tag{3.4}$$

where K_0 is the Macdonald function and the parameter κ will be specified later. Since K_0 is strictly decreasing, the corresponding ψ_α will not be smooth at $r = R$ but it remains continuous, hence it is an admissible trial function. To estimate the first term on the right-hand side of (3.2), let us compute

$$K_0(\kappa R)^2 \int_{\mathbb{R}^2} |f'_{R,\kappa}(r)|^2\, \mathrm{d}^2x = 2\pi \int_{\kappa R}^\infty K_1(t)^2\, t\, \mathrm{d}t$$
$$= \pi\left[\kappa^2 R^2 K_1'(\kappa R)^2 - \left(\kappa^2 R^2 + 1\right) K_1(\kappa R)^2\right]$$

cf [AS, equation 9.6.26], [PBM, equation 1.12.3.2]. Using $-K_1'(\xi) = K_0(\xi) + \xi^{-1}K_1(\xi)$ in combination with the asymptotic expressions $K_0(\xi) = -\ln\xi + \mathcal{O}(1)$, $K_1(\xi) = \xi^{-1} + \mathcal{O}(\ln\xi)$ for $\xi \to 0$, we find that

$$\|f'_{R,\kappa}\|^2_{L^2(\mathbb{R}^2)} < -\frac{C}{\ln(\kappa R)} \tag{3.5}$$

holds for a positive constant C and κR small enough. This makes it possible to estimate the first term on the right-hand side of (3.2) using the fact that the functions χ_j are bounded outside $\mathcal{B}_R$; recall that $\chi_j(x) = \mathcal{O}(|x|^{-F+j})$ at large distances and $F - j \geqslant 0$.

We will show that $\int_\Sigma B(x)\left|\sum_j \alpha_j\chi_j(x)\right|^2 \mathrm{d}^2x > 0$ also holds in this situation again by assuming the opposite. Indeed, let us set $h_j := h\chi_j$ with a real-valued $h \in C_0^2(\Sigma)$ in (3.1); then the fourth term on the right-hand side of (3.2) takes the form

$$2\varepsilon \int_\Sigma \left|\sum_{j=0}^{[F]} \alpha_j\chi_j(x)\right|^2 h(x)B(x)\, \mathrm{d}^2x.$$

Since B is non-zero, it is possible to choose h so that the integral is strictly negative. Taking ε positive and small enough the sum of the last four terms on the right-hand side of (3.2) with $\mu = 2$ can be made negative, since the the linear (in ε) term prevails over the quadratic ones and the third term is supposedly non-positive. The first term

$$\int_{\mathbb{R}^2} \left|f'_{R,\kappa}(r)\right|^2 \left|\sum_{j=0}^{[F]} \alpha_j \chi_j(x)\right|^2 \mathrm{d}^2 x$$

is positive, but the χ_j's are bounded outside $\mathcal{B}_R$ and $|\alpha| = 1$, so it can be made sufficiently small by a suitable choice of κ. Again using the supersymmetry property, $D^*D + 2B = DD^*$, we arrive at the absurd conclusion that $\|D^*\psi_\alpha\|^2 < 0$.

Hence we can finally take the trial functions (3.1) with $f_{R,\kappa}$ given by (3.4) and $\varepsilon = 0$, which yields the estimate

$$(\psi_\alpha, H_P^{(-)}(A)\psi_\alpha) < -\frac{C}{\ln(\kappa R)} \max_{0 \leqslant j \leqslant [F]} \|\chi_j\|_\infty^2 - \frac{1}{2}(g-2) \min_{|\alpha|=1} \left(\int_\Sigma B(x) |\psi_\alpha(x)|^2 \, \mathrm{d}^2 x \right). \tag{3.6}$$

The second term on the right-hand side is strictly negative if $g > 2$, since $\int_\Sigma B(x)\, |\psi_\alpha(x)|^2 \, \mathrm{d}^2 x > 0$ for any α in a compact set (surface of the hypersphere $|\alpha| = 1$), and it dominates the sum for κ small enough.

To conclude the proof of the first claim, one has to check that the trial functions (3.1) indeed span a subspace of dimension $1 + [F]$. This follows readily from the linear independence of $\psi_j := f_{R,\kappa}\chi_j$, $j = 0, \ldots, [F]$; recall that the χ_j's are linearly independent and coincide with ψ_j at least in the set $\mathcal{B}_R$.

If $F = 0$, the function $\widetilde{\chi}_0(x) := \mathrm{e}^{\phi(x)}$ which solves $D^*\widetilde{\chi}_0 = 0$ is also bounded at large distances and we can apply the analogous argument to the operator $H_P^{(+)}(A) = DD^* + \frac{1}{2}(g-2)B$. Using a properly chosen function $\widetilde{\psi}_0 = f_{R,\kappa}\widetilde{\chi}_0 + \varepsilon h$, we can show that $\int_\Sigma B(x)|\widetilde{\chi}_0(x)|^2 \, \mathrm{d}^2 x < 0$, so $(\widetilde{\psi}_0, H_P^{(+)}(A)\widetilde{\psi}_0) < 0$ for $g > 2$ and small κ; hence $H_P^{(+)}(A)$ also has a bound state. □

Remarks. (a) The argument fails only if $B = 0$, since then $\chi_0(x) = \widetilde{\chi}_0(x) = 1$ and the matrix elements $\langle B \rangle_{\chi_0}$ and $\langle B \rangle_{\widetilde{\chi}_0}$ are zero.
(b) Instead of the tail modification (3.4), a simpler one ($f_R(r) := f(r/R)$, where $f \in C_0^\infty(\mathbb{R}_+)$ is such that $f(u) = 0$ for $u \geqslant 2$) was employed in [BEZ2]. The kinetic energy is in this case estimated by

$$\frac{1}{R^2} \int_{\mathbb{R}^2} \left| f'\left(\frac{r}{R}\right)\right|^2 \left|\sum_{j=0}^{[F]} \alpha_j \chi_j(x)\right|^2 \mathrm{d}^2 x \leqslant C \|f'\|_\infty^2 R^{-2(F-[F])}$$

for a positive C. It is clear that one can handle the whole problem in this way, except for the case of integer F.

4. Weakly bound states in two dimensions

Schrödinger operators in dimensions one and two can have bound states for arbitrarily weak potentials, so the behaviour of the ground state in these cases is of particular interest. The corresponding asymptotic formulae, known already to Landau and Lifshitz [LL], were analysed rigorously in [Si, BGS, Kl]. If we digress from the subject here, it is because we want to draw attention to interesting aspects of the case when the interaction is switched off in a nonlinear way.

It is sufficient, of course, to describe peculiarities of the nonlinear case. Thus consider a two-dimensional Schrödinger operator family

$$H(\lambda) = -\Delta + V(\lambda, x) \tag{4.1}$$

on $L^2(\mathbb{R}^2)$ with λ belonging to an interval $[0, \lambda_0]$, where the potentials satisfy

$$V(\lambda, x) = \lambda V_1(x) + \lambda^2 V_2(x) + W(\lambda, x) \tag{4.2}$$

with

$$|W(\lambda, x)| \leqslant \lambda^3 V_3(x) \tag{4.3}$$

and

$$V_j \in L^{1+\delta}(\mathbb{R}^2) \cap L(\mathbb{R}^2, (1 + |x|^\delta)\, \mathrm{d}^2x) \qquad j = 1, 2, 3 \tag{4.4}$$

for some $\delta > 0$. By the Birman–Schwinger principle, (4.1) has an eigenvalue $\epsilon(\lambda) = -\kappa^2$ iff the integral operator K_κ with the kernel

$$K_\kappa^\lambda(x, y) = |V(\lambda, x)|^{1/2}\, R_0(\kappa; x, y)\, V(\lambda, y)^{1/2}$$

(where $V^{1/2} := |V|^{1/2} \operatorname{sign} V$) has an eigenvalue -1; here

$$R_0(\kappa; x, y) = \frac{1}{2\pi}\, K_0(\kappa|x - y|) \tag{4.5}$$

is the kernel of the free resolvent $(-\Delta + \kappa^2)^{-1}$. A standard trick is then to split the operator under consideration into two parts, $K_\kappa^\lambda = L_\kappa^\lambda + M_\kappa^\lambda$, where the former is rank-one with the kernel $L_\kappa^\lambda(x, y) = -\frac{1}{2\pi}\, |V(\lambda, x)|^{1/2} \ln\kappa\, V(\lambda, y)^{1/2}$, while the latter is regular as $\kappa \to 0+$ and, in this limit, has kernel

$$M_0^\lambda(x, y) = -\frac{1}{2\pi}\, |V(\lambda, x)|^{1/2} \left\{\gamma + \ln\frac{|x - y|}{2}\right\} V(\lambda, y)^{1/2}$$

where γ is the Euler constant. Now we employ the identity

$$(I + K_\kappa^\lambda)^{-1} = \left[I + (I + M_\kappa^\lambda)^{-1} L_\kappa^\lambda\right]^{-1} (I + M_\kappa^\lambda)^{-1}$$

where the existence of the inverses on the right-hand side for sufficiently small λ follows from the assumptions made about the potential in the same way as in [Si]. The spectral problem is thus reduced to finding a singularity of the square bracket, which leads to an implicit equation. If we put $u := (\ln\kappa)^{-1}$, it can be written as

$$u - \frac{1}{2\pi} \int V(\lambda, x)^{1/2}\, (I + M_\kappa^\lambda)^{-1}(x, y)\, |V(\lambda, y)|^{1/2}\, \mathrm{d}^2x\, \mathrm{d}^2y = 0 \tag{4.6}$$

and used to derive the Taylor expansion of the function $\lambda \mapsto u(\lambda)$; a weakly bound state with the eigenvalue $\epsilon(\lambda) = -\mathrm{e}^{2/u(\lambda)}$ exists iff $u(\lambda) < 0$ about the origin.

In the linear case, $V = \lambda V_1$, from here we get the usual expansion

$$u(\lambda) = \frac{\lambda}{2\pi} \int V_1(x)\, \mathrm{d}^2x + \frac{\lambda^2}{4\pi^2} \int V_1(x) \left\{\gamma + \ln\frac{|x - y|}{2}\right\} V_1(y)\, \mathrm{d}^2x\, \mathrm{d}^2y + \mathcal{O}(\lambda^3) \tag{4.7}$$

which shows that a bound state exists iff $\int V_1(x)\, \mathrm{d}^2x \leqslant 0$ (the second term is negative if the potential V_1 is non-trivial and has zero mean [Si]).

For a potential family (4.2) nonlinear in λ the sign of $\int V_1(x)\, \mathrm{d}^2x$ is again decisive. An interesting situation arises, however, if the linear part has zero mean:

$$\int V_1(x)\, \mathrm{d}^2x = 0. \tag{4.8}$$

Replacing λV_1 with (4.2) in (4.6) and expanding in powers of λ we find that

$$u(\lambda) = \lambda^2 \left\{ \frac{1}{2\pi} \int V_2(x)\, \mathrm{d}^2 x + \frac{1}{4\pi^2} \int V_1(x)\, \ln|x-y|\, V_1(y)\, \mathrm{d}^2 x\, \mathrm{d}^2 y \right\} + \mathcal{O}(\lambda^3) \tag{4.9}$$

holds in this case (the term with $\gamma - \ln 2$ splits into a product of one-dimensional integrals and vanishes, too). We arrive at the following conclusion.

Proposition 1. *An operator family (4.1) with the potential satisfying (4.2)–(4.4) and (4.8) has a weakly bound state provided the leading coefficient in (4.9) is negative. If it is positive, no bound state exists for small* λ*.*

The formula (4.9) also yields the asymptotic behaviour of the corresponding eigenvalue $\epsilon(\lambda) = -\mathrm{e}^{2/u(\lambda)}$. We will not inquire about the critical case when the second-order coefficient also vanishes.

5. The centrally symmetric case

Let us return now to the Pauli operator (2.1) and consider the situation when the field is centrally symmetric, so the vector potential can be chosen in the symmetric gauge, $\boldsymbol{A}(x) = \lambda A(r)\boldsymbol{e}_\varphi$, with $A(r) = r^{-1}\int_0^r B(r')\, r'\, \mathrm{d}r'$. We have introduced the positive parameter λ in order to discuss how the spectral properties depend on the field strength. We can perform a partial-wave decomposition and replace (2.1) by the family of operators

$$H_\ell^{(\pm)}(\lambda) = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + V_\ell^{(\pm)}(\lambda, r) \qquad V_\ell^{(\pm)}(\lambda, r) := \left(\lambda A(r) + \frac{\ell}{r}\right)^2 \pm \frac{\lambda}{2}\, g B(r) \tag{5.1}$$

on $L^2(\mathbb{R}^+, r\, dr)$. In [BEZ1] these operators were used to discuss the behaviour of an electron in the magnetic field induced by a localized rotating electric current. We need not insist on that here, assuming only that the field is locally integrable with $B(r) = \mathcal{O}(r^{-2-\delta})$ as $r \to \infty$. However, we will be interested primarily in the typical situation for current-induced magnetic fields, in which the field has zero mean (i.e. $F = 0$) since the flux lines are closed in $\mathbb{R}^3$.

It was shown in [BEZ1] under stronger assumptions (involving a smoothness and a faster decay of the field) that each orbital Hamiltonian $H_\ell^{(-)}(\lambda)$ has a bound state for λ large enough, the critical values for emergence of these states being, of course, ℓ-dependent. This result relies only on the behaviour of $V_\ell^{(-)}(\lambda, r)$ about the origin and is thus independent of the fact that $F = 0$, the important point being that $g > 2$ so the ground-state energy of the harmonic oscillator obtained in the limit $\lambda \to \infty$ is negative, cf [BEZ1].

The 'spin-up' Pauli operator $H_P^{(+)}(\lambda)$ may exhibit less intuitive behaviour as suggested by theorem 1. If $F = 0$ for a *compactly supported* field, then $H_P^{(+)}(\lambda)$ has also a bound state for any $\lambda > 0$. Recall that theorem 1 says nothing about the size of Σ, it may be quite large. Inspecting the shape of the effective potentials $V_\ell^{(\pm)}(\lambda, r)$ for the two cases we see that the states with different spin orientations are supported in different regions: 'spin-down' states in the vicinity of the origin (out of the centrifugal barrier for $\ell \neq 0$), while the 'spin-up' state at large distances where (for an arbitrary but fixed λ) the magnetic field term dominates slightly over the quadratic one in $V_0^{(+)}(\lambda, r)$, creating a shallow potential well.

Let us examine the *weak-coupling behaviour* in the case of a vanishing total flux, $F = 0$, in the field with a tail, $B(r) = \mathcal{O}(r^{-2-\delta})$; no smoothness assumption is made. If $\ell \neq 0$, the first term in $V_\ell^{(\pm)}(\lambda, r)$ is bounded below by $\lambda v(r)$ for a suitably chosen positive function v with compact support (the simplest choice is $v(r) = c\,\Theta(r_0 - r)$ for appropriate c and r_0). Since the second term does not contribute to $\int_0^\infty V_\ell^{(\pm)}(\lambda, r)\, r\, \mathrm{d}r$ which determines the linear part of the weak-coupling behaviour, it follows from (4.7) and the minimax principle that the discrete spectrum of $H_\ell^{(\pm)}(\lambda)$ is empty for λ small enough, when the centrifugal barrier prevents binding.

The interesting case is, of course, the s-wave part, where the effective potential acquires the form (4.2) with the last term absent and

$$V_1(r) = \pm\frac{g}{2}\,B(r) \qquad V_2(r) = A(r)^2. \tag{5.2}$$

In view of the assumption about the field, $r \mapsto A(r)$ is absolutely continuous and $\mathcal{O}(r^{-1-\delta})$, so the condition (4.4) is satisfied. It remains to evaluate the second integral in (4.9): we have

$$\frac{1}{4\pi^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} V_1(x)\,\ln|x-y|\,V_1(y)\,\mathrm{d}^2x\,\mathrm{d}^2y$$
$$= \frac{g^2}{4}\frac{1}{2\pi}\int_0^\infty \mathrm{d}r\,r B(r)\int_0^\infty \mathrm{d}r'\,r' B(r')\int_0^{2\pi}\ln[r^2 + r'^2 - 2rr'\cos\varphi]^{1/2}\,\mathrm{d}\varphi.$$

By [GR, 4.224.9] the last integral equals $2\pi\ln\max(r, r')$; we substitute this into the formula and integrate repeatedly by parts using $rB(r) = (rA(r))'$. This yields

$$-\frac{g^2}{4}\int_0^\infty\left(\int_r^\infty A(r')\,\mathrm{d}r'\right)B(r)\,r\,\mathrm{d}r = -\frac{g^2}{4}\int_0^\infty \mathrm{d}r\,A(r)^2\,r\,\mathrm{d}r.$$

Thus we arrive at the following conclusion.

Proposition 2. *Let a spherically symmetric magnetic field B be locally integrable with $B(r) = \mathcal{O}(r^{-2-\delta})$ and vanishing flux, $F = 0$. Then each of the operators $H_0^{(\pm)}(\lambda)$ with $g > 2$ has a negative eigenvalue for sufficiently small λ.*

Remarks. (a) The relation (4.9) yields also the asymptotic behaviour of the bound state energy,

$$\epsilon^{(\pm)}(\lambda) \approx -\exp\left\{-\left(\frac{\lambda^2}{8}(g^2-4)\int_0^\infty A(r)^2\,r\,\mathrm{d}r\right)^{-1}\right\} \tag{5.3}$$

as $\lambda \to 0$ where $\approx$ has the usual meaning (cf [Si]). The leading term is thus the same for both spin orientations; however, since $g \neq 2$, the second theorem of [AC] does not apply and the degeneracy may be lifted in the next order.
(b) Notice that the argument of the previous section cannot be applied to compactly supported fields with a non-zero flux, since the corresponding vector potential then has too slow a decay, $A(r) = \mathcal{O}(r^{-1})$, and consequently $V_1 \notin L(\mathbb{R}^2, (1+|x|^\delta)\,\mathrm{d}^2x)$. One may ask whether the asymptotics is nevertheless $\epsilon(\lambda) \approx \exp(-4/\lambda Fg)$ as follows from a formal application of (4.7). The example worked out in [CFC] leads to the conclusion that it is not the case (see equation (17) of that paper). The question about the asymptotic behaviour thus remains open.
(c) Another open question is whether the bound state of $H_0^{(\pm)}(\lambda)$ survives generally for λ large if the field is not compactly supported.

6. Non-symmetric weak coupling revisited

By different means, the result of the previous section complements the zero-flux part of theorem 1 in the weak-coupling case. While imposing the symmetry requirement, it relaxes the assumptions on the field decay. Here we want to show that the above argument can be carried through for non-symmetric fields as well under mild regularity assumptions; the price we shall pay is to have a weaker form of the asymptotic formula (5.3) only. Specifically, suppose that

$$|B(x)| \leqslant C_1\langle x\rangle^{-2-\delta} \qquad \int_{\mathbb{R}^2}\frac{|B(y)|}{|x-y|}\,\mathrm{d}^2y \leqslant C_2\langle x\rangle^{-1-\delta} \tag{6.1}$$

where $\langle x\rangle := \sqrt{1+r^2}$. Then we have the following result.

Proposition 3. *Let a magnetic field B satisfy the conditions (6.1) for some $C_1, C_2, \delta > 0$, and let $F = 0$. Then each of the operators $H_P^{(\pm)}(\lambda)$ with $g > 2$ has a negative eigenvalue for λ small enough.*

Proof. This is based on two observations. The first concerns the 'mixed' term $2\mathrm{i}A\cdot\nabla$ in the Hamiltonian; we shall show that it does not contribute to the energy form for real-valued functions (the other 'mixed' term, $\mathrm{i}\nabla\cdot A$, vanishes in the gauge we have been adopting). More specifically, take a real-valued $\psi \in C_0^2(\mathbb{R}^2)$, i.e. twice differentiable with compact support. For the sake of brevity, we write the vector potential components as $A_i = -\epsilon_{ij}\partial_j\phi$, where ϵ is the two-dimensional Levi–Civita tensor, and employ the convention of summation over repeated indices; then

$$
\begin{aligned}
(\psi, A\cdot\nabla\psi) &= -\int_{\mathbb{R}^2}\psi(x)\,\epsilon_{ij}\,(\partial_j\phi)(x)\,(\partial_i\psi)(x)\,\mathrm{d}^2x \\
&= -\frac{1}{2}\lim_{R\to\infty}\int_{\mathcal{B}_R}\epsilon_{ij}\,(\partial_j\phi)(x)\,(\partial_i\psi^2)(x)\,\mathrm{d}^2x \\
&= -\frac{1}{2}\lim_{R\to\infty}\int_{\mathcal{B}_R}\epsilon_{ij}\left\{(\partial_j(\phi\,\partial_i\psi^2))(x) - \phi(x)\,(\partial_i\partial_j\psi^2)\right\}\mathrm{d}^2x \\
&= \frac{1}{2}\lim_{R\to\infty}\oint_{\partial\mathcal{B}_R}\phi(x)\,(\nabla\psi^2)(x)\cdot\mathrm{d}\boldsymbol{\ell}(x) = 0.
\end{aligned}
$$

The third line is obtained from the second using integration by parts. Its second term vanishes because $\partial_j\partial_i\psi^2$ is symmetric with respect to the interchange of indices and is contracted with the anti-symmetric symbol ϵ_{ij}. The remaining term is rewritten by means of the Stokes theorem and vanishes in the limit since $\nabla\psi^2$ has compact support.

The second observation is that the relation between the two integrals which we found in the proof of proposition 2 by explicit computation in polar coordinates is valid generally. To see this, let us rewrite $\int A(x)^2\,\mathrm{d}^2x$ by means of the first Green identity:

$$
\begin{aligned}
\int_{\mathbb{R}^2} A(x)^2\,\mathrm{d}^2x &= \lim_{R\to\infty}\int_{\mathcal{B}_R}(\nabla\phi(x))^2\,\mathrm{d}^2x \\
&= \lim_{R\to\infty}\int_{\mathcal{B}_R}\left\{(\nabla\cdot(\phi\nabla\phi))(x) - \phi(x)\,(\nabla^2\phi)(x)\right\}\mathrm{d}^2x \\
&= \lim_{R\to\infty}\oint_{\partial\mathcal{B}_R}\phi(x)\,(\nabla\phi)(x)\cdot\mathrm{d}\boldsymbol{\sigma}(x) - \lim_{R\to\infty}\int_{\mathcal{B}_R}\phi(x)B(x)\,\mathrm{d}^2x.
\end{aligned}
\tag{6.2}
$$

In the second integral we have used $\triangle\phi = B$, and the first one was rewritten by means of the Gauss theorem. Our aim is now to use conditions (6.1) to demonstrate that the first integral on the right-hand side vanishes in the limit. The decay hypothesis about the field yields

$$
\begin{aligned}
|\phi(x)| &\leqslant \frac{1}{2\pi}\int_{|y-x|\leqslant 1}|B(y)|\,|\ln|x-y||\,\mathrm{d}^2y + \frac{1}{2\pi}\int_{|y-x|\geqslant 1}|B(y)|\,\ln|x-y|\,\mathrm{d}^2y \\
&\leqslant \frac{C_1}{2\pi}\int_{|z|\leqslant 1}|\ln|z||\,\mathrm{d}^2z + \frac{C_1}{2\pi}\int_{|z|\geqslant 1}\langle x-z\rangle^{-2-\delta}\,\ln|z|\,\mathrm{d}^2z.
\end{aligned}
$$

Denote $\ln_+ u := \max(0, \ln u)$. Then for any $\eta > 0$ there is a $K_\eta > 0$ such that

$$
\ln_+|z| < K_\eta\,\langle x-z\rangle^\eta\,(1 + \ln_+|x|).
$$

This follows from the fact that

$$\limsup_{\zeta,\xi\to\infty} \frac{\ln_+ \zeta}{\left(1+|\xi-\zeta|^2\right)^{\eta/2}(1+\ln_+\xi)} \leqslant 1$$

in the first quadrant, and the function under the limit is continuous there. The second of the above integrals is thus estimated by

$$\frac{C_1 K_\eta}{2\pi}\,(1+\ln_+|x|)\int_{\mathbb{R}^2}\langle y\rangle^{-2-\delta+\eta}\,\mathrm{d}^2 y.$$

For $\eta < \delta$ the last integral is convergent; hence there is a $C' > 0$ such that $|\phi(x)| \leqslant C' \ln|x|$ as $|x| \to \infty$. The second one of the conditions (6.1) implies

$$|A(x)| = |(\nabla\phi)(x)| \leqslant \frac{C_2}{2\pi}\langle x\rangle^{-1-\delta}$$

so the first integral on the right-hand side of (6.2) is $\mathrm{o}(R^{-\delta'})$ for any $\delta' < \delta$ and vanishes in the limit that we have set out to prove. Substituting (2.3) for ϕ in the second integral, we finally arrive at the identity

$$\int_{\mathbb{R}^2} A(x)^2\,\mathrm{d}^2x = -\frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} B(x)\,\ln|x-y|\,B(y)\,\mathrm{d}^2x\,\mathrm{d}^2y. \qquad (6.3)$$

The conditions (6.1) ensure that both integrals exist. Now it is easy to conclude the proof. We have

$$\begin{aligned}\inf\sigma\left(H_P^{(\pm)}(\lambda)\right) &= \inf\left\{\left(\psi, H_P^{(\pm)}(\lambda)\psi\right) :\ \psi\in D\left(H_P^{(\pm)}(\lambda)\right)\right\}\\ &\leqslant \inf\left\{\left(\psi, H_P^{(\pm)}(\lambda)\psi\right) :\ \psi\in C_0^2(\mathbb{R}^2),\ \psi=\bar\psi\right\}\\ &= \inf\sigma\left(\widetilde{H}_P^{(\pm)}(\lambda)\right) \qquad (6.4)\end{aligned}$$

where

$$\widetilde{H}_P^{(\pm)}(\lambda) := -\Delta + \lambda^2 A(x)^2 \pm \tfrac{1}{2}\lambda g B(x).$$

The last equality in (6.4) is due to the fact that $C_0^2(\mathbb{R}^2)$ is a core of $\widetilde{H}_P^{(\pm)}(\lambda)$. It is now sufficient to apply proposition 1 to the operator $\widetilde{H}_P^{(\pm)}(\lambda)$ and to employ the identity (6.3). □

Remark. In view of the estimate used in the proof, relation (5.3) is now replaced by the asymptotic inequality

$$\epsilon^{(\pm)}(\lambda) \lesssim -\exp\left\{-\left(\frac{\lambda^2}{16\pi}\,(g^2-4)\int_{\mathbb{R}^2} A(x)^2\,\mathrm{d}^2x\right)^{-1}\right\}. \qquad (6.5)$$

The question as to whether the right-hand side is still the lower bound remains open.

Note added in proof. In a recent paper [W] Weidl discusses the discrete spectrum coming from matrix-potential perturbations of the Aharonos–Casher states.

Acknowledgments

The authors acknowledge the hospitality extended to them at the Institutes where parts of the work were done: CPT Luminy (PE) and NPI Řež (VZ). The research was partially supported by GACR and GAAS under contracts 202/96/0218 and 1048801. RMC was supported by CNPq and NSF under grant No PHY94-07194.

References

[AC] Aharonov Y and Casher A 1979 Ground state of a spin-1/2 charged particle in a two-dimensional magnetic field *Phys. Rev.* A **19** 2641–2

[AGHH] Albeverio S, Gesztesy F, Høegh-Krohn R and Holden H 1988 *Solvable Models in Quantum Mechanics* (Berlin: Springer)

[AS] Abramowitz M S and Stegun I A (ed) 1965 *Handbook of Mathematical Functions* (New York: Dover)

[BEZ1] Bentosela F, Exner P and Zagrebnov V A 1998 Electron trapping by a current vortex *J. Phys. A: Math. Gen.* **31** L305–11

[BEZ2] Bentosela F, Exner P and Zagrebnov V A 1999 Anomalous electron trapping by magnetic flux tubes and electric current vortices *Mathematical Results in Quantum Mechanics (Operator Theory: Advances and Applications)* ed J Dittrich (Basel: Birkhäuser) pp 191–6

[BGS] Blanckenbecler R, Goldberger M L and Simon B 1977 The bound states of weakly coupled long-range one-dimensional quantum Hamiltonians *Ann. Phys., NY* **108** 69–78

[BV] Bordag M and Voropaev S 1993 Charged particle with magnetic moment in the Aharonov-Bohm potential *J. Phys. A: Math. Gen.* **26** 7637–49

[CC] Cavalcanti R M and de Carvalho C A A 1998 Bound states of a spin-1/2 charged particle in a magnetic flux tube *J. Phys. A: Math. Gen.* **31** 7061–3

[CFC] Cavalcanti R M, Fraga E S and de Carvalho C A A 1997 Electron localization by a magnetic vortex *Phys. Rev.* B **56** 9243–6

[GR] Gradshtein I S and Ryzhik I M 1980 *Tables of Integral, Series, and Products* (New York: Academic)

[Kl] Klaus M 1977 On the bound state of Schrödinger operators in one dimension *Ann. Phys., NY* **108** 288–300

[LL] Landau L D and Lifshitz E M 1965 *Quantum Mechanics: Non-Relativistic Theory* (Oxford: Pergamon)

[Mo] Moroz A 1996 Single-particle density of states, bound states, phase-shift flip, and a resonance in the presence of an Aharonov–Bohm potential *Phys. Rev.* A **53** 669–94

[PBM] Prudnikov A P, Brychkov Yu O and Marichev O I 1986 *Integrals and Series, II. Special Functions* (New York: Gordon and Breach)

[Si] Simon B 1976 The bound state of weakly coupled Schrödinger operators in one and two dimensions *Ann. Phys., NY* **97** 279–88

[Th] Thaller B 1992 *The Dirac Equation* (Berlin: Springer)

[W] Weidl T 1999 Remarks on virtual bound states for semibounded operators *Commun. PDE* **24** 25–60